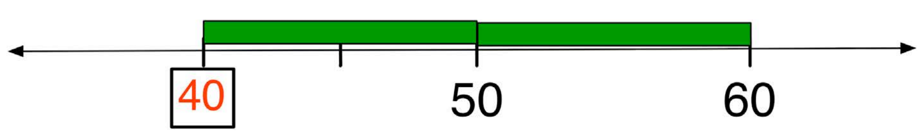
Learning Mathematics through Representations

Integers and Fractions on the Number Line

A curriculum unit for 4th and 5th grade

Write the number that belongs in the box.



Geoffrey B. Saxe, Principal Investigator
University of California, Berkeley

Funded by the Institute of Education Sciences 2007-2013

The *Learning Mathematics through Representations* curriculum was developed at the University of California, Berkeley by a collaborative team with diverse expertise. The contributors were:

Faculty
Geoffrey B. Saxe, Principal Investigator
Maryl Gearhart

Doctoral students
Ronli Diakow
Darrell Earnest
Lina Chopra Haldar
Bona Kang
Katherine Lewis
Anna McGee
Yasmin Sitabkhan
Ying Zheng

Teachers
Rick Kleine
Jennifer Pfotenhauer

Faculty consultants
Deborah Loewenberg Ball, University of Michigan
Hyman Bass, University of Michigan

Additional contributors to LMR research were:

Faculty
Sophia Rabe-Hesketh

Doctoral students
Nicole Leveille Buchanan
Jennifer Collett
David Torres Irribarra
Kenton de Kirby
Marie Le
Amanda McKerracher
Meghan Shaughnessy
Alison Miller Singley
Joshua Sussman

Learning Mathematics through Representations was supported in part by the Institute of Education Sciences grants R305B070299 (research grant) and R305B090026 (predoctoral training grant) to the University of California, Berkeley. The opinions expressed are those of the authors and do not represent views of the Institute of Education Sciences or the U.S. Department of Education. We are grateful to the teachers and students who participated in our research.

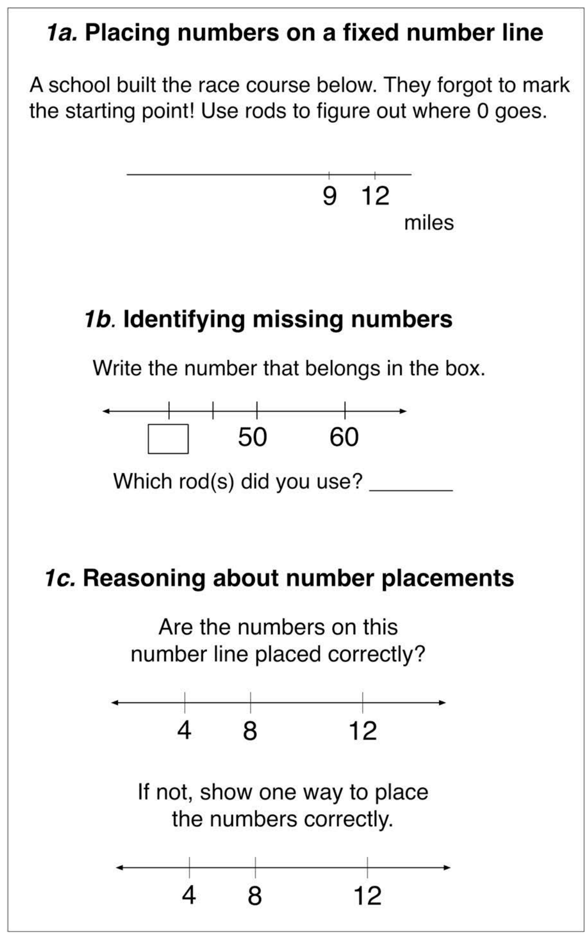
Introduction

The *Learning Mathematics through Representations* curriculum focuses on core ideas about the representation of number on the number line. As students progress through the curriculum, they develop rich understandings of integers and fractions as they solve number line problems and justify their solutions. LMR's strategy is to engage students with explicitly defined mathematical ideas as resources for mathematical argumentation and justification, an instructional approach recommended by the *Common Core State Standards*.

> *One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from (Common Core State Standards Initiative, 2010).*

LMR students work with ideas such as order and unit interval as they solve non-routine problems like the ones in Figure 1. Tasks like these were developed through a process of design research to ensure that tasks were accessible and yet challenging for a broad range of 4th and 5th grade students. Figure 1a engages students in locating 0 on a line marked with two fixed integers, a challenging task for students accustomed to conventional number lines with 0 marked; students must use the marked interval of three units to measure the distance from 9 to 0. To determine the missing value in 1b, students must reason about relationships between interval lengths and tick mark values to determine that the solution is 40 (not 30, the solution if students skip count by 10s along the tick marks). To evaluate whether numbers are placed correctly on the line in 1c, students must recognize that the skip counting pattern (4, 8, 12) is not correct because the interval distances between the numbers are not consistent, even if the numbers are skip counting by 4s.

Figure 1. Sample tasks



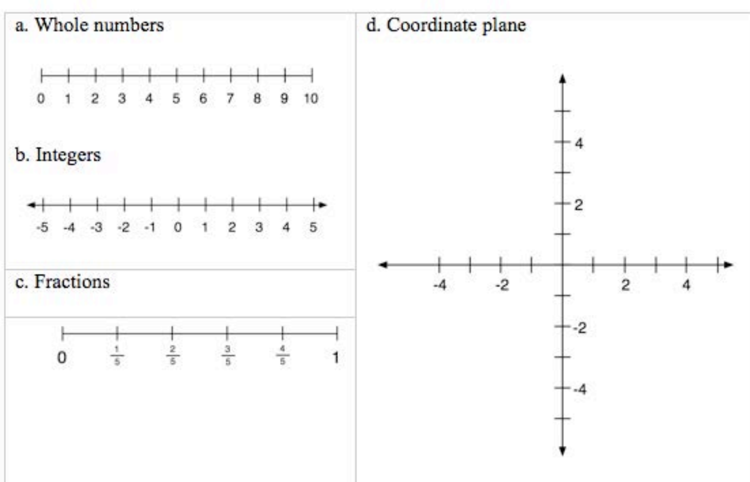
Why integers & fractions, and what is the role of the number line?

The U.S. mathematics curriculum has often been characterized as "a mile wide and an inch deep" (Schmidt, McKnight & Raizen, 1997[1]). Teachers are required to teach so much content that the coherence of the mathematics is compromised. The topics of integers and fractions, for example, are often treated as unrelated when they should be unified within an integrated framework for rational numbers. It is no wonder that students come away with partial and disconnected understandings of the mathematics.

The LMR curriculum provides a coherent treatment of mathematically related content strands.

- *Integers and fractions:* LMR lessons provide opportunities for students to develop and enrich their understandings of core ideas about number. Students first construct understandings of integers on the number line, and then build on that understanding to reason about fractions on the number line. By using the number line as the central representation, the lessons foster understanding of inherent connections between integers and fractions.
- *Number line:* The number line itself is a core representation in mathematics. The number line is woven throughout the elementary, middle school and high school curriculum, and, in the professional world, it is a core representation in mathematics and science. Figure 2 illustrates number line representations that are common in the elementary and middle school curricula. Cultivating a rich understanding of the number line should be an important instructional objective.

Figure 2. Number lines in elementary and middle school curriculum



[1] Schmidt, W. H., McKnight, C. C., & Raizen, S. A. (1996). *A splintered vision: An investigation of U.S. science and mathematics education.* Dordrecht: Kluwer Academic Publishers.

A research-based curriculum

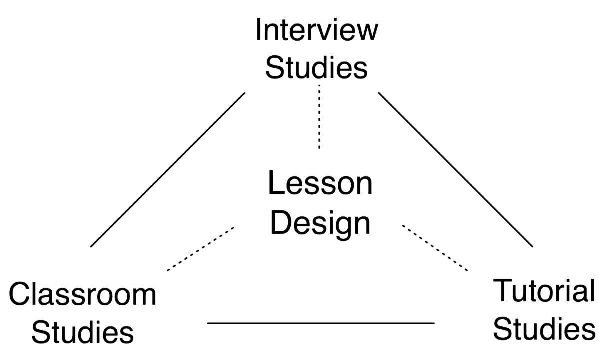
The authors of the *Common Core State Standards* (2010) argue that mathematics standards and curriculum materials must be research-based.

> *[Needed are] research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill and understanding develop over time.*

The LMR project has produced research findings that support the mission of the Common Core Standards Initiative.

In the initial phase of the project, the LMR team conducted three types of studies in 4th and 5th grade classrooms to identify how students reason about the representation of integers and fractions on the number line, and the pedagogical techniques that support students' insights and progress (see Figure 3). *Interview* studies documented students' understandings of core number line ideas, like unit and multiunit intervals, and the role of 0. *Tutorial* studies helped us identify instructional methods that support student insight and progress. Through *classroom studies,* we developed lessons in collaboration with expert teachers.

Figure 3. Design research: Interview, tutorial and classroom studies



In the final phase of the project, we conducted an efficacy study that contrasted the learning gains in 11 LMR and 10 matched comparison classrooms. The performance of both groups was identical at pretest, and all students made strong gains over the academic year; however, the gains of LMR students were 75% greater than the gains of comparison students on both the end-of-LMR and the end-of year assessments. These findings provide support for the efficacy of the LMR curriculum in strengthening 4^{th} and 5^{th} students' understandings of integers and fractions.

What teachers should know

To support teachers as they implement LMR lesson materials, each lesson guide begins with a page entitled "what teachers should know." Here we introduce the LMR curriculum using the same organization:

- *Objectives* describe what students should know by the end of the lesson.
- *About the Math* provides background on the lesson's mathematical content.
- *About Student Understanding* is an orientation to the kinds of "partial understandings" that students bring to the lesson as they make sense of lesson tasks and discussions.
- *About the Pedagogy* is an overview of pedagogical strategies recommended for the lesson.

Objectives

As students work with the LMR curriculum, they will develop deeper understandings of integers and fractions, and learn to use the number line as a powerful mathematical representation. LMR objectives cover the core ideas for positive and negative integers (e.g., order, unit interval, the symmetry of positive and negative integers around 0), the core ideas for fractions less than and greater than 1, equivalent fractions, and ordering and comparing fractions. Students also develop practices of mathematical argumentation as they reason about solutions to number line problems.

About the Math

Anatomy of the Number Line

Figure 4 depicts the mathematical ideas that are the foundation of the LMR curriculum, and shows how the number line is a geometric representation of number. *Order* and *interval* are central ideas.

Order. Order includes relations of 'less than' and 'greater than.' These ideas are expressed on the number line as spatial direction: Numbers increase in value from left to right and decrease in value from right to left.

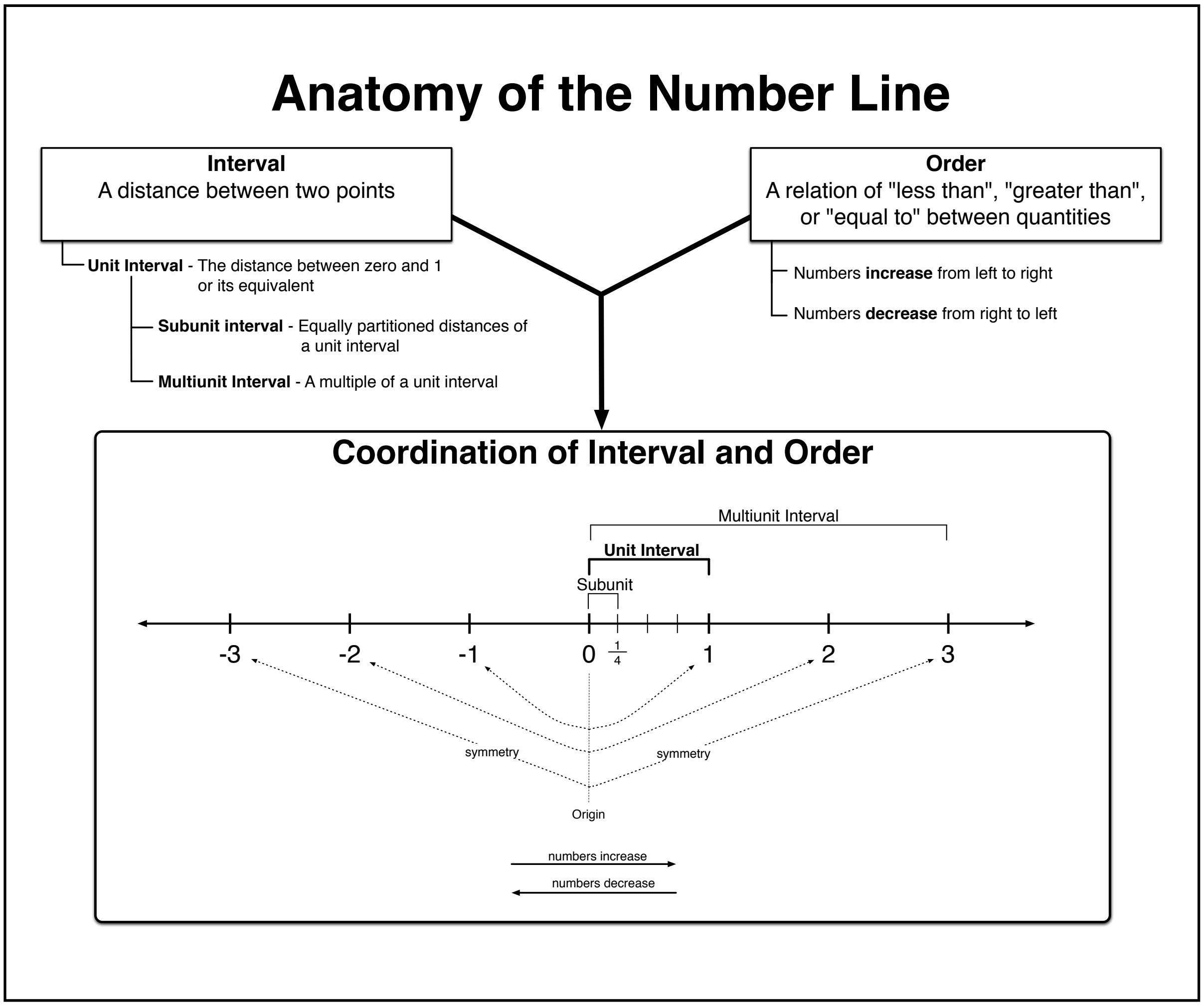
Interval. Once two points on the number line are labeled, a numerical interval is defined, and every number has a place. Three kinds of intervals on the line are the target of LMR lessons.

- *Unit intervals* are the distance between 0 and 1 or its equivalent distance. Unit intervals are the basis for locating integers on the number line;
- *Subunit intervals* are equally partitioned distances of a unit interval and the basis for locating fractions on the line.
- *Multiunit intervals* are multiples of unit intervals and the basis for locating numbers in a patterned multiunit series (e.g., by 2s, 3s, 10s, 100s, etc.).

The numerical label for a point on a number line represents its distance from zero, a distance understood in terms of units, subunits, and/or multiunits.

Understanding the representation of positive and negative integers on the number line also involves the idea of symmetry. The number line is *symmetric* in that, for every positive number, there is a negative number that is the same distance from zero. The positions of the positives and negatives are thus mirror images of one another.

Figure 4. Anatomy of the number line



Over the course of the lessons, students investigate the ideas represented in Figure 4 as well as entailments of these ideas. For example, students investigate the conjecture that "every number has a place on the number line, but not every number needs to be shown," a conjecture that builds on the core definitions for unit interval and order.

Principles and Definitions

The LMR lesson sequence is designed so that the big ideas in each lesson build on prior lessons. The ideas are expressed as number line "*Principles and Definitions"* that are introduced to open up topics for inquiry and help students communicate mathematical ideas and support mathematical arguments. As the lessons progress, the teacher records each new principle on a large class poster, and students record them on their own copies. The collection of principles and definitions is a shared resource for mathematical work and discussion.

Figures 5A and 5B display the lesson sequence for integers and fractions respectively. Figures 6A and 6B outline the sequence of Principles and Definitions for integers and fractions.

- *Positive Integers (Positive Integers Lessons 1-8)*. This unit covers positive integers. Over the course of 7 lessons, the principles of ***order***, ***0 is a number***, ***unit interval***, ***interval***, and ***multiunit interval*** are introduced. Lesson 8 is a review of Positive Integers.
- *Negative Integers (Negative Integers Lessons 1-4)*. These four lessons cover negative integers through the introduction of the ***symmetry*** principle. These lessons also engage students in applying the principles for positive integers to negative integers, and this is a challenge for many students. For example, students grapple with the idea that the number -4 is less than the number 1 because it is to the left of 1, and, according to the order principle, numbers decrease from right to left.
- *Fractions (Fractions Lessons 1-14)*. In the 14 LMR fractions lessons, students build on the number line principles and definitions for integers. In Lessons 1-8, students investigate part-whole relationships with the support of the definition for ***subunit***, the definition of ***fraction*** in terms of subunits, and the extension of these ideas to ***mixed numbers*** and ***fractions greater than 1***. In Lessons 9-13, students investigate numerator-denominator relationships with support of additional definitions for ***equivalence*** and ***benchmarks*** as they solve problems with equivalent fractions, ordering, and comparing. Lesson 14 is a review of Fractions.

Figure 5A. Integers scope and sequence

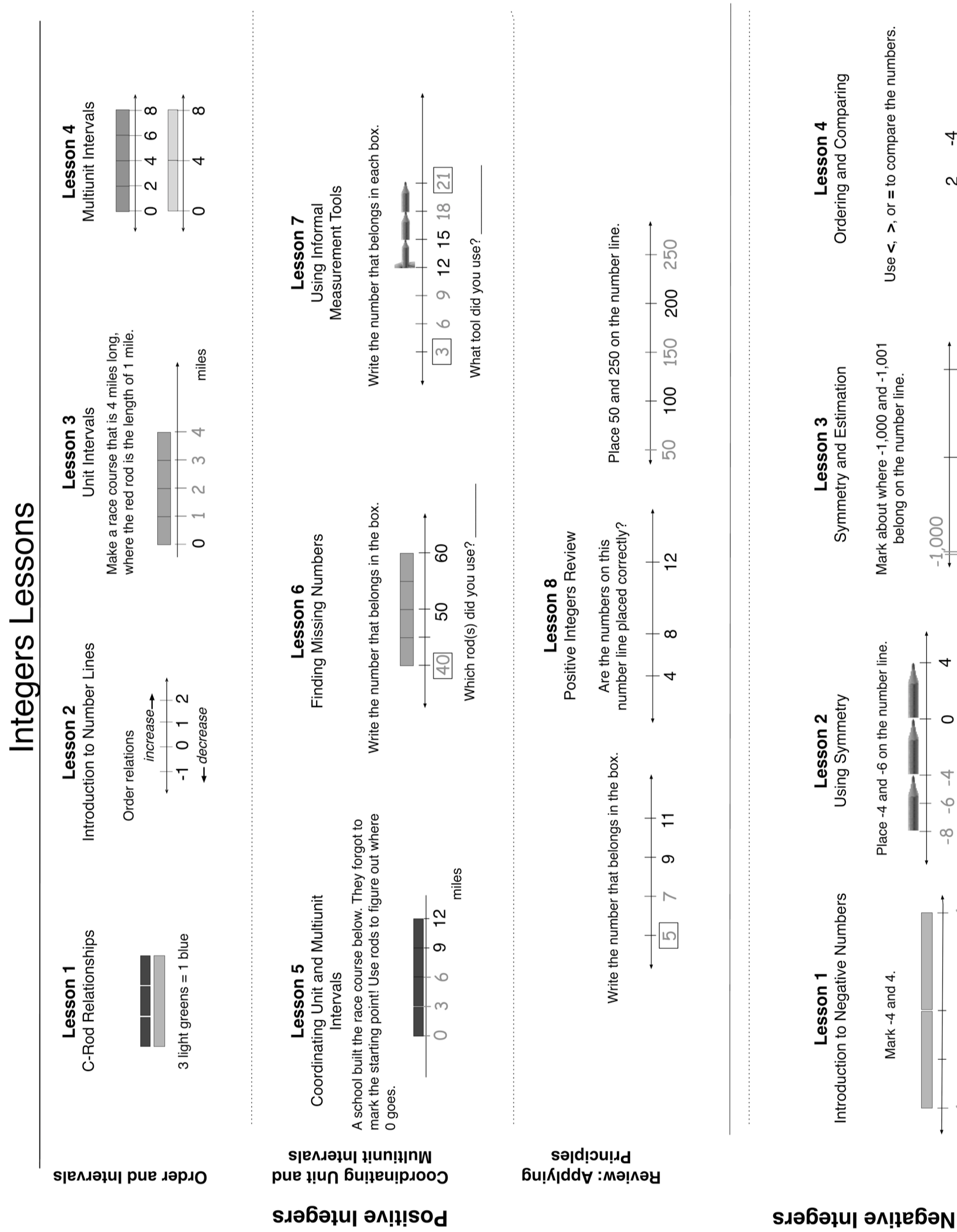


Figure 5B. Fractions scope and sequence

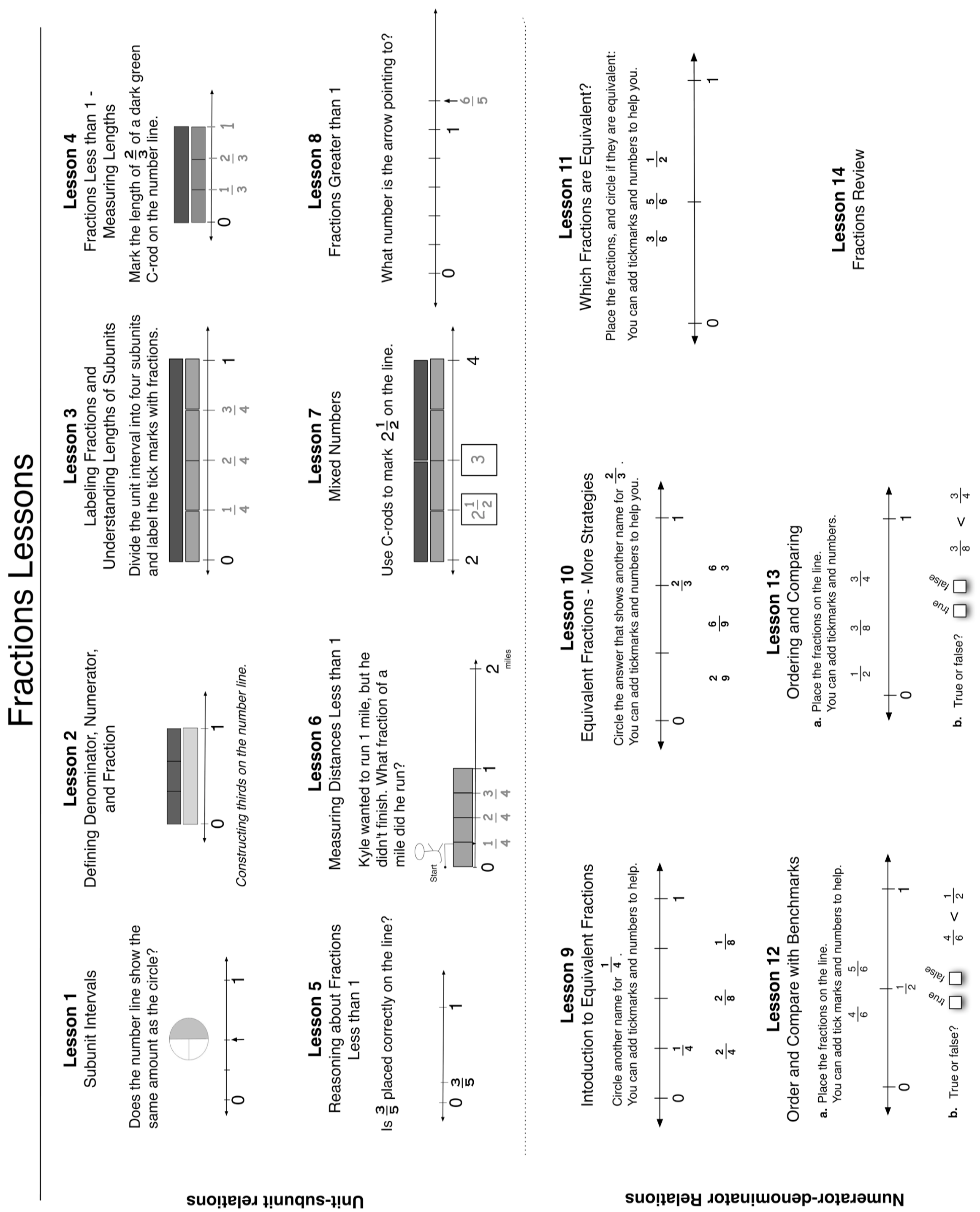


Figure 6A. Integers Principles and Definitions

Number Line Principles and Definitions

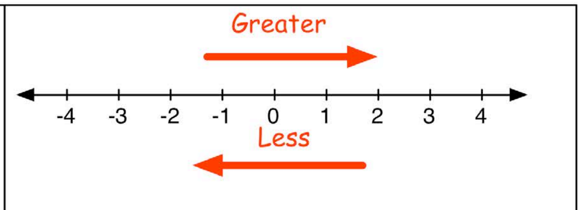
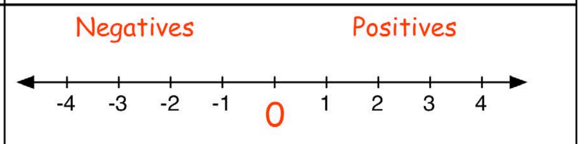
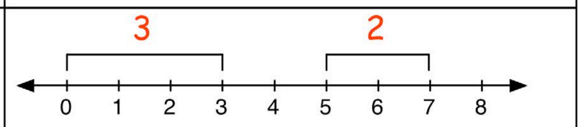
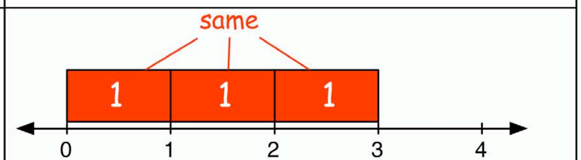
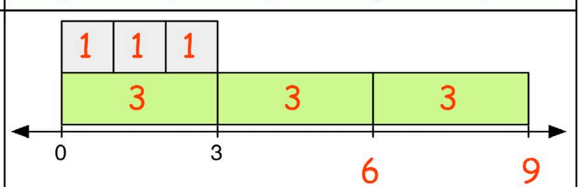
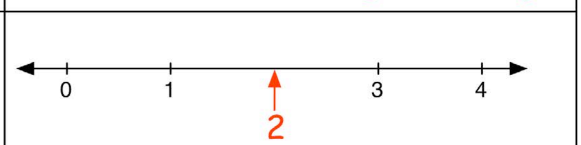
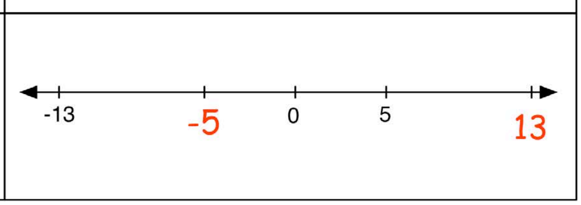
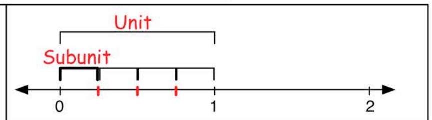
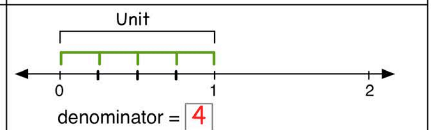
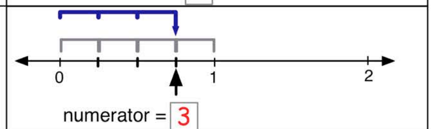
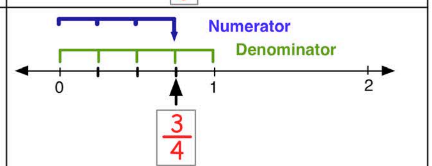
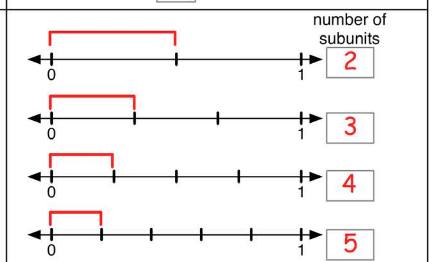
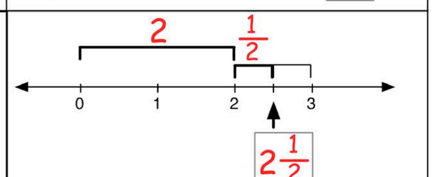
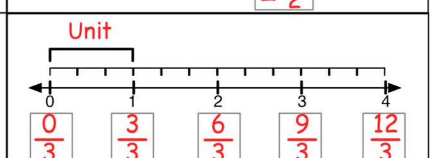
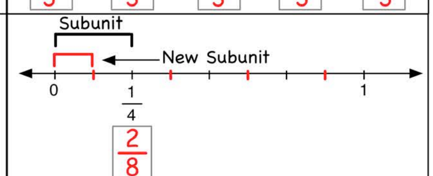
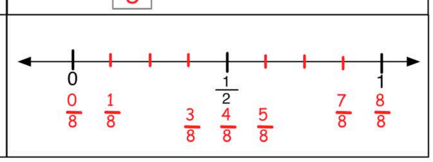
Name	Definition/Principle	Example
Order	Numbers increase in value from left to right. Numbers decrease from right to left. Numbers at the same place have same value.	
0 is a number	0 is a number, so it has a place on the number line.	
Interval	The distance between any two numbers on the number line.	
Unit Interval	A unit interval is the distance from 0 to 1 or any distance of 1.	
Multiunit Interval	A multiple of a unit interval.	
Every number has a place	Every number has a place on the line, but not all need to be shown.	
Symmetry	For every positive number, there is a negative number that is the same distance from 0.	

Figure 6B. Fractions Principles and Definitions

Number Line Principles and Definitions

Name	Definition/Principle	Example
Subunit	Dividing a unit into equal distances creates subunits.	
Denominator	The number of subunits in a unit.	
Numerator	The number of subunits.	
Fraction	numerator / denominator	
Length of the subunit	The more subunits in a unit the shorter the subunits are.	
Mixed Number	A whole number and a fraction.	
Whole Numbers as Fractions	A whole number can be written as a fraction	
Equivalent Fractions	Fractions that are in the same place but with different subunits.	
Benchmarks	0, 1/2 and 1 are benchmarks that are useful for ordering and comparing fractions.	

About Student Understandings

When fourth- and fifth-grade students begin the LMR curriculum, they bring prior knowledge that serves as a resource for reasoning and problem-solving. Students have knowledge of:

- greater than and less than relationships for whole numbers;
- counting, skip counting, adding, subtracting with whole numbers;
- area models of fractions, including finding equivalent fractions by re-partitioning;
- linear measurement tools, such as rulers and thermometers.

While these understandings are resources, they are insufficient as a foundation for understanding integers and fractions on the number line. For example, students with knowledge of order relations may use a counting strategy to solve the non-routine task in Figure 1a—they may count backwards from 9 by 1s, and become confused when the line appears to be too short to place 0. Other students with knowledge of the conventions of the number line may place 0 at the leftmost point without considering the lengths of unit or multiunit intervals.

In LMR lessons, students build upon what they know as they construct or revise mathematical principles and definitions, and coordinate principles to locate, order, and compare numbers on the number line.

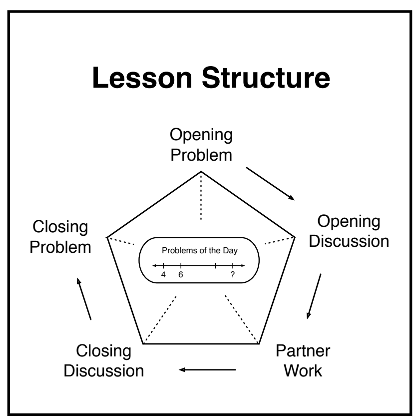
About the Pedagogy

Lesson Design

LMR lessons have the 5-phase structure illustrated in Figure 7.

- *Opening Problems* establish the focus of the day's lesson. These problems are generally non-routine tasks (see Figure 1) that elicit a diversity of student ideas.
- *Opening Discussion* provides a context for surfacing the diversity of students' ideas and reasoning about conflicts among ideas.
- *Partner Work* engages students in building on the ideas in the Opening Discussion. Students work in pairs to solve variations of the Opening Problems.
- *Closing Discussion* is the context for resolving remaining contradictions and reviewing key ideas.
- *Closing Problems* serve as formative assessment of students' progress.

Figure 7. LMR lesson design



Instructional Methods

Three design principles guided the design of the instructional methods in each phase of the lesson.

- *Communication: eliciting student thinking and building on student ideas, affording students opportunities to contribute mathematical ideas;*

  The non-routine tasks featured in the Opening Problems and Partner Work are designed to elicit a range of student ideas. In the Opening Discussion, Partner Work, and Closing Discussion, students have opportunities to communicate their diverse ideas. Principles & Definitions and Cuisenaire™ rods (and informal measurement tools) support reasoning and communication.

- *Connections: provoking cognitive conflict and engaging students in analysis and resolution, affording students opportunities to make connections among mathematical ideas;*

  Discrepancies among students' solutions and reasoning create the need for mathematical resolution. The guide suggests ways to guide students toward resolution through the application of Principles & Definitions and the use of measurement tools to represent relationships among interval lengths, interval values, and tick mark values. Discussions incorporate techniques to focus students on contrasting ideas. For example, if no student has expressed a partial understanding, a teacher can "push student thinking" by introducing an imaginary student who expresses a partial understanding.

- *Equity: differentiating instruction in shared mathematical contexts, affording all students opportunities to engage with core mathematical content.*

  LMR lessons are designed to provide all students the opportunity to engage with LMR's core curricular content in a shared mathematical context. Through our initial research, we established that LMR tasks are accessible and yet challenging for a wide range of students in heterogeneous urban classrooms. To engage all students with the mathematics, discussions use techniques to help students feel safe when sharing or commenting on mathematical thinking—for example, think-pair-share or discussion of an imaginary student's reasoning. Partner Work is organized as a sequence of tasks that increase in challenge.

Table 1. Role of LMR design principles in each lesson phase

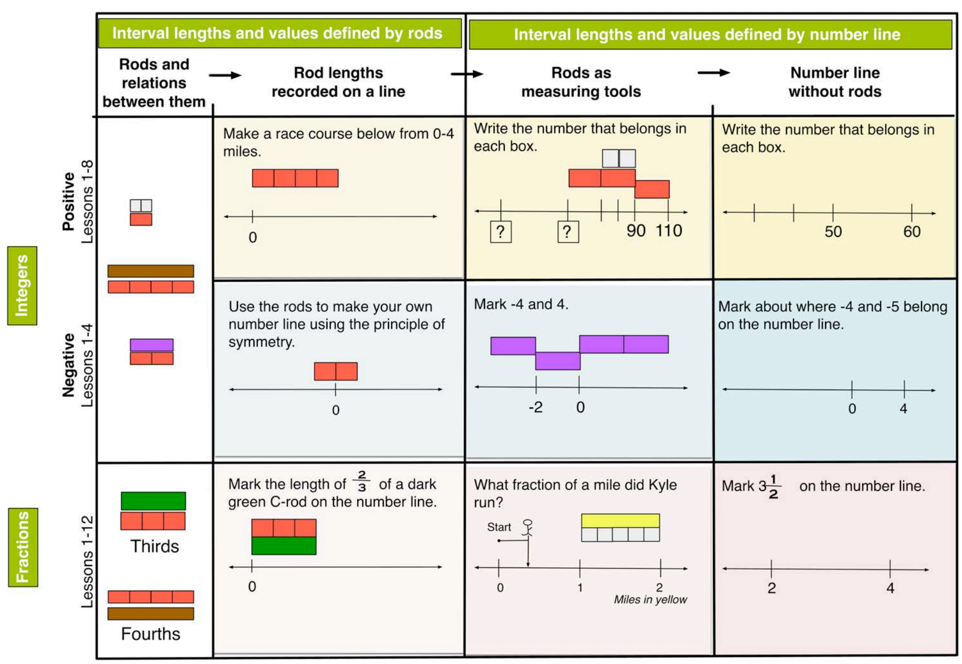
	Communication: Eliciting and building on student thinking, affording students opportunities to contribute mathematical ideas	**Connection:** Provoking and resolving cognitive conflict, affording students opportunities to make connections among mathematical ideas	**Equity:** Differentiating instruction in shared mathematical contexts, affording all students opportunities to engage with core content
Opening Problems	The Opening Problems engage students with the focus of the day's lesson. As students solve the problems, they make sense of them in terms of prior knowledge. Students will do this in different ways, and they will have the opportunity to share their ideas during the Opening Discussion.	The Opening Problems are non-routine tasks that lead students to conflicts about solutions. Students' dilemmas create the need for a mathematical resolution. During the lesson, students resolve dilemmas with Number Line Principles & Definitions.	Opening Problems that focus on 'big ideas' orient all students to the day's lesson. Our studies have demonstrated that these tasks are accessible and challenging for a wide range of students. Students' diverse ideas about the problems are shared during Discussions and Partner Work.
Opening Discussion	Students present and justify their diverse ideas during the Opening Discussion. Teachers can surface these ideas through: - prompts and questions, - asking students to draw and discuss their solutions to opening problems, and - making use of CuisenaireTM rods for unitizing and for making thinking public The Guide provides resources on patterns of student thinking that may surface. The teacher guides students to organize their mathematical ideas in reference to Principles & Definitions. Classroom norms encourage the sharing of diverse ideas that are valued for discussion and learning.	In the Opening Discussion, the teacher guides students to identify contradictions by: - posing non-routine tasks (e.g., "crazy number lines"), - "pushing student thinking" by introducing alternative ideas, - encouraging students to share diverse ideas, and - drawing attention to contrasting ideas. The teacher then guides students to resolve contradictions by making connections between mathematical ideas and between these ideas and the Number Line Principles & Definitions.	Students should be encouraged to share their diverse ideas during the Opening Discussion. Techniques include: - providing safe contexts for sharing (e.g., Think-Pair-Share, followed by whole-class share out); - establishing classroom norms of respect for all mathematical ideas; - introducing an imaginary student when "pushing student thinking;" - asking students to use whiteboard C-rods to make their reasoning public; - posting Principles & Definitions for reference.
Partner Work	Partners make their reasoning public using C-rods and Principles & Definitions as resources for problem-solving and discussion. As partners listen to and observe each other, they can build upon one another's reasoning.	Non-routine tasks in the Partner Work pose contradictions. Partners resolve contradictions and disagreements by connecting their ideas to Principles & Definitions and by using C-rods as resources to justify solutions.	Partners work together in different ways – sharing ideas, collaborating on strategies, assisting. Worksheets (a) engage all students in the same tasks (b) while providing additional challenging tasks. When advanced students work on extra worksheets, the teacher focuses on students who need support.
Closing Discussion	Teachers surface student ideas in the Closing Discussion through: - teacher questions linked to Principles & Definitions, - student presentations, - think-pair-share activities followed by whole-class discussion of partner ideas Discussion of ideas is similar to that in the Opening Discussion, but the focus shifts to making connections between task solutions, students' mathematical ideas, and Principles & Definitions.	In the Closing Discussion, students resolve conflicts through the application of Number Line Principles & Definitions. The Guide resources include: - samples of common student responses; - "pushing student thinking" by introducing an imaginary student with alternative ideas; - prompts and questions to help students focus on connecting ideas to shared Principles & Definitions.	The Closing Discussion integrates students' diverse ideas. Techniques include: - providing safe contexts for sharing (e.g., Think-Pair-Share); - establishing classroom norms of respect for all mathematical ideas; - introducing an imaginary student when - - "pushing student thinking;" - asking students to use whiteboard C-rods to make their reasoning public; - posting Principles & Definitions for reference.
Closing Problems	Closing Problems serve as assessments of student learning and progress. Non-routine tasks pose contradictions and elicit different patterns of student thinking.	Students' solutions to the Closing Problems are evidence of students' progress in identifying and resolving contradictions, using Principles & Definitions as resources.	All students work on the same Closing Problems. Sorting students' Closing Problems serves as a formative assessment, and the teacher then works with students who need additional support.

Cuisenaire™ rods and Informal Measurement Tools

In LMR lessons, Cuisenaire™ rods support reasoning and communication about unit intervals, multiunit intervals, and subunit intervals on the number line. As outlined in Figure 8, there is a sequential pattern to the role of these measurement tools within each chunk of the curriculum

- *Rods as physical magnitudes recorded on a line:* As shown in the first column of Figure 9, in the introduction of each topic, rods are treated as physical magnitudes off the number line. Students use rods to construct a linear distance or a rod relationship, and then record their model on a line with only a single number marked (typically zero).
- *Rods as measuring tools:* When given a number line with two numbers marked, students can use rods as measuring tools to determine the lengths and values of the marked interval, and to locate or identify other numbers on the line. Students are also encouraged to use informal measurement tools such as pencil lengths to support a more generalized understanding of relationships among interval lengths, interval values, and the number value assigned to each tick mark.
- *Reasoning without rods:* Eventually students are posed problems that cannot be solved with rods, because rods do not fit the marked distances on these tasks. As students use informal measurement tools or 'eyeball' interval values, they reason about interval values, interval relationships, and the locations of numbers on the line.

Figure 8. Unitizing with Cuisenaire™ rods



Features of the lesson guides

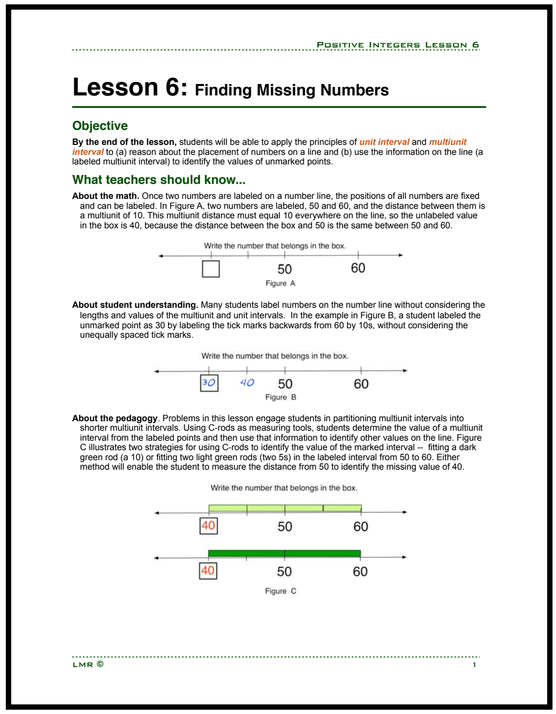
This section provides an overview of the features of the lesson guides.

Lesson Background

Pages 1-3 of each lesson guide provide background.

Page 1 provides information on the *Objective* and *What teachers should know:*

- *About the math*
- *About student understanding*
- *About the pedagogy*



POSITIVE INTEGERS LESSON 6

Lesson 6: Finding Missing Numbers

Objective

By the end of the lesson, students will be able to apply the principles of ***unit interval*** and ***multiunit interval*** to (a) reason about the placement of numbers on a line and (b) use the information on the line (a labeled multiunit interval) to identify the values of unmarked points.

What teachers should know...

About the math. Once two numbers are labeled on a number line, the positions of all numbers are fixed and can be labeled. In Figure A, two numbers are labeled, 50 and 60, and the distance between them is a multiunit of 10. This multiunit distance must equal 10 everywhere on the line, so the unlabeled value in the box is 40, because the distance between the box and 50 is the same between 50 and 60.

Figure A

About student understanding. Many students label numbers on the number line without considering the lengths and values of the multiunit and unit intervals. In the example in Figure B, a student labeled the unmarked point as 30 by labeling the tick marks backwards from 60 by 10s, without considering the unequally spaced tick marks.

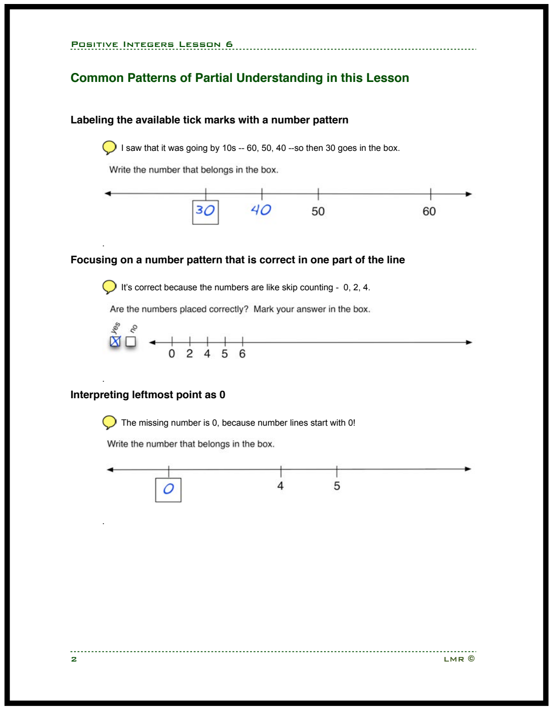
Figure B

About the pedagogy. Problems in this lesson engage students in partitioning multiunit intervals into shorter multiunit intervals. Using C-rods as measuring tools, students determine the value of a multiunit interval from the labeled points and then use that information to identify other values on the line. Figure C illustrates two strategies for using C-rods to identify the value of the marked interval -- fitting a dark green rod (a 10) or fitting two light green rods (two 5s) in the labeled interval from 50 to 60. Either method will enable the student to measure the distance from 50 to identify the missing value of 40.

Figure C

LMR © 1

Page 2 provides information on the common patterns of partial student understanding that are likely to emerge throughout the lesson.



POSITIVE INTEGERS LESSON 6

Common Patterns of Partial Understanding in this Lesson

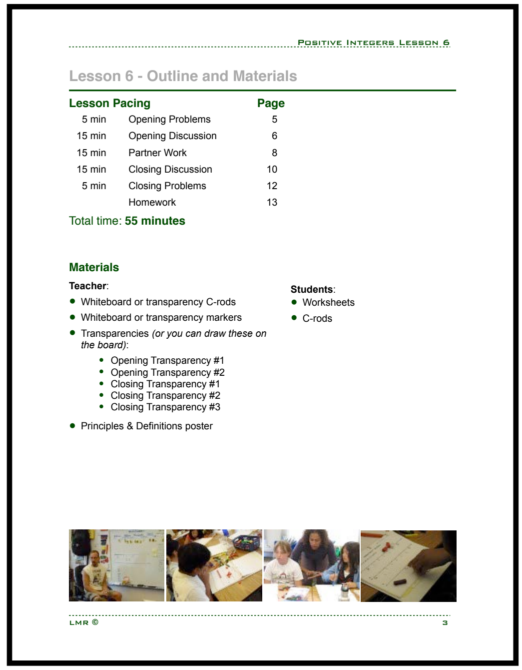
Labeling the available tick marks with a number pattern

Focusing on a number pattern that is correct in one part of the line

Interpreting leftmost point as 0

2 LMR ©

Page 3 is an overview of the lesson phases with estimated timing, page numbers, and materials needed. If relevant, this page also identifies any new Principles and Definitions to be introduced in the lesson.



Positive Integers Lesson 6

Lesson 6 - Outline and Materials

Lesson Pacing		Page
5 min	Opening Problems	5
15 min	Opening Discussion	6
15 min	Partner Work	8
15 min	Closing Discussion	10
5 min	Closing Problems	12
	Homework	13

Total time: **55 minutes**

Materials

Teacher:

- Whiteboard or transparency C-rods
- Whiteboard or transparency markers
- Transparencies *(or you can draw these on the board)*:
  - Opening Transparency #1
  - Opening Transparency #2
  - Closing Transparency #1
  - Closing Transparency #2
  - Closing Transparency #3
- Principles & Definitions poster

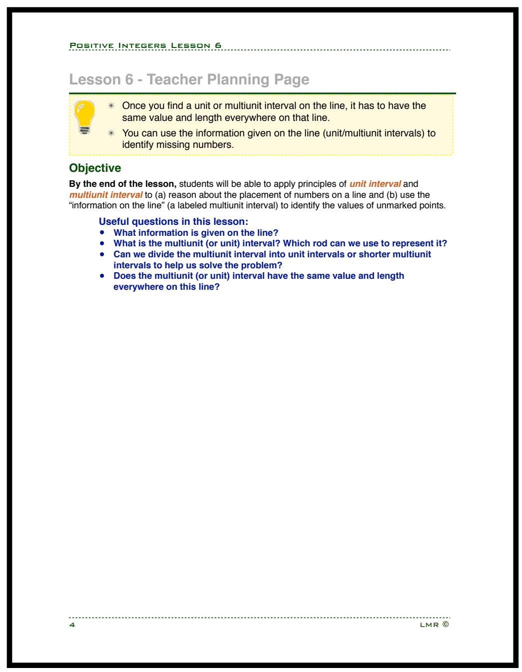
Students:

- Worksheets
- C-rods

LMR © 3

.

Page 4 is a page for teachers to take notes and outline their lesson plan. The page highlights the big ideas, objective, and useful prompts for supporting student reasoning.



Positive Integers Lesson 6

Lesson 6 - Teacher Planning Page

- Once you find a unit or multiunit interval on the line, it has to have the same value and length everywhere on that line.
- You can use the information given on the line (unit/multiunit intervals) to identify missing numbers.

Objective

By the end of the lesson, students will be able to apply principles of *unit interval* and *multiunit interval* to (a) reason about the placement of numbers on a line and (b) use the "information on the line" (a labeled multiunit interval) to identify the values of unmarked points.

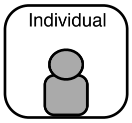
Useful questions in this lesson:

- **What information is given on the line?**
- **What is the multiunit (or unit) interval? Which rod can we use to represent it?**
- **Can we divide the multiunit interval into unit intervals or shorter multiunit intervals to help us solve the problem?**
- **Does the multiunit (or unit) interval have the same value and length everywhere on this line?**

4 LMR ©

Guide Features

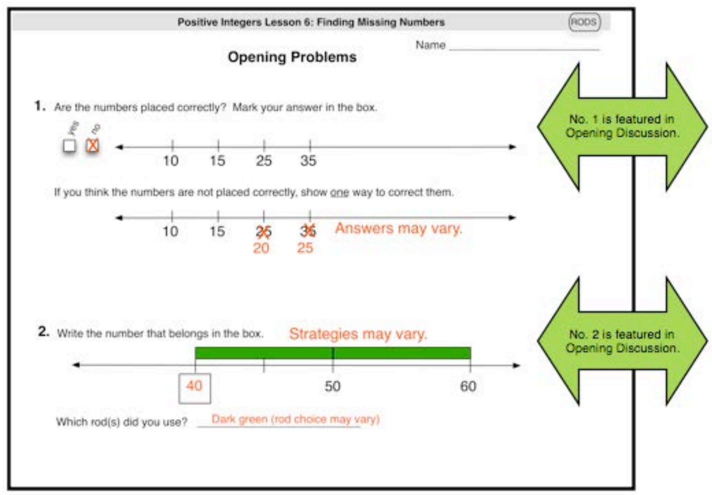
The body of each guide provides support for each phase of the lesson. Icons cue particular resources or methods.

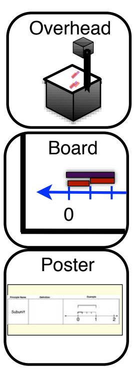


Opening Problems: Most lessons begin with independent work on non-routine Opening Problems. These tasks set up contradictions that challenge students' understandings of the representation of number on the number line. The teacher roves to notice the range in students' ideas.

Lesson Guide features include:

- answer keys (although correct answers and strategies may vary)
- green arrows to indicate which problems will be the focus of the Opening Discussion

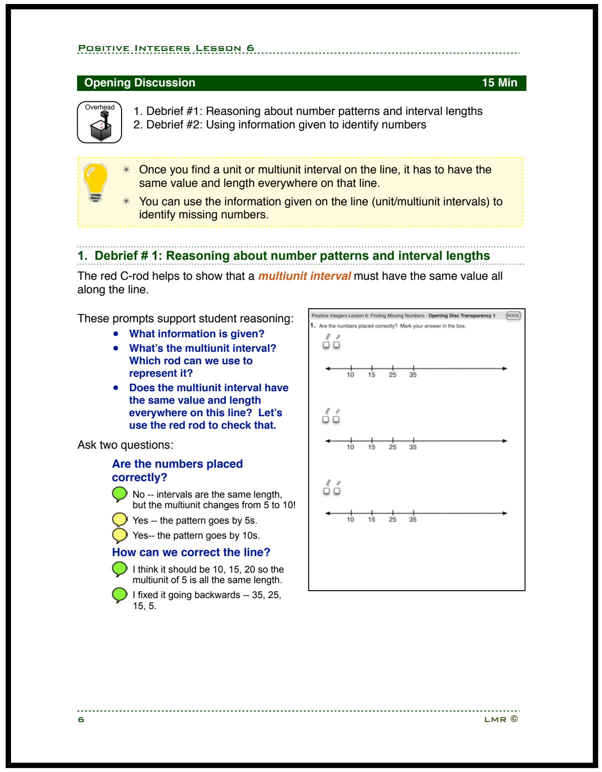




Opening Discussion: In the Opening Discussion, the teacher supports students as they communicate their mathematical ideas about selected Opening Problems. Students represent their solution strategies, and reason with the support of principles and definitions, rods, and informal measurement tools.

Teachers may choose to conduct the discussion at the whiteboard (by re-creating the task) or around the overhead projector using the provided sheets.

- The guide identifies the big ideas of each Opening Discussion with a light bulb icon in yellow.
- The optional transparency is shown as an inset.
- Blue text represents useful discussion prompts.
- Talk bubbles represent common student ideas: Green bubbles illustrate more complete understandings, and yellow bubbles illustrate typical partial understandings.



Positive Integers Lesson 6

Opening Discussion 15 Min

Overhead

1. Debrief #1: Reasoning about number patterns and interval lengths
2. Debrief #2: Using information given to identify numbers

- Once you find a unit or multiunit interval on the line, it has to have the same value and length everywhere on that line.
- You can use the information given on the line (unit/multiunit intervals) to identify missing numbers.

1. Debrief # 1: Reasoning about number patterns and interval lengths

The red C-rod helps to show that a *multiunit interval* must have the same value all along the line.

These prompts support student reasoning:

- **What information is given?**
- **What's the multiunit interval? Which rod can we use to represent it?**
- **Does the multiunit interval have the same value and length everywhere on this line? Let's use the red rod to check that.**

Ask two questions:

Are the numbers placed correctly?

- No -- intervals are the same length, but the multiunit changes from 5 to 10!
- Yes -- the pattern goes by 5s.
- Yes-- the pattern goes by 10s.

How can we correct the line?

- I think it should be 10, 15, 20 so the multiunit of 5 is all the same length.
- I fixed it going backwards -- 35, 25, 15, 5.

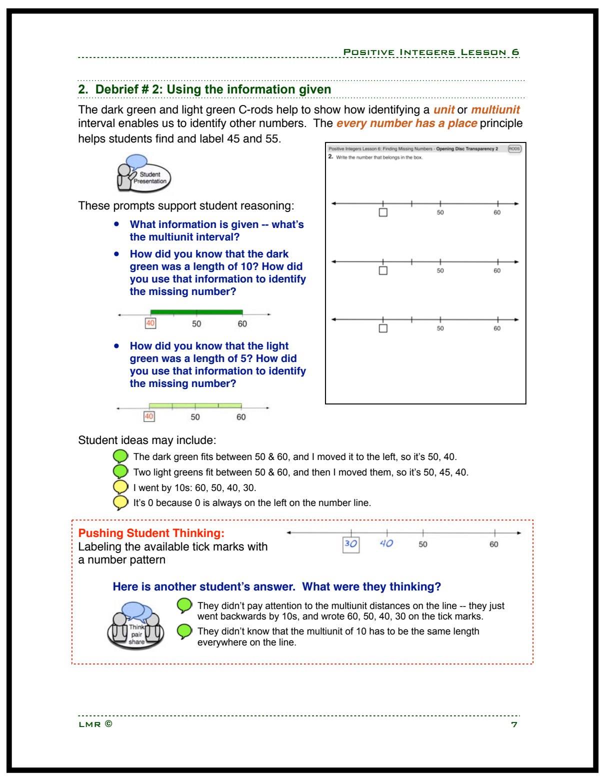
10 15 25 35

10 15 25 35

10 15 25 35

6 LMR ©

The page at right illustrates two of the icons for suggested participation formats -- student presentations (top of the page) and think-pair-share (bottom of the page).



Positive Integers Lesson 6

2. Debrief # 2: Using the information given

The dark green and light green C-rods help to show how identifying a *unit* or *multiunit* interval enables us to identify other numbers. The *every number has a place* principle helps students find and label 45 and 55.

Student Presentation

These prompts support student reasoning:

- **What information is given -- what's the multiunit interval?**
- **How did you know that the dark green was a length of 10? How did you use that information to identify the missing number?**

40 50 60

- **How did you know that the light green was a length of 5? How did you use that information to identify the missing number?**

40 50 60

50 60

50 60

50 60

Student ideas may include:

- The dark green fits between 50 & 60, and I moved it to the left, so it's 50, 40.
- Two light greens fit between 50 & 60, and then I moved them, so it's 50, 45, 40.
- I went by 10s: 60, 50, 40, 30.
- It's 0 because 0 is always on the left on the number line.

Pushing Student Thinking:
Labeling the available tick marks with a number pattern

30 40 50 60

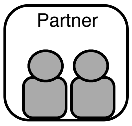
Here is another student's answer. What were they thinking?

Think pair share

- They didn't pay attention to the multiunit distances on the line -- they just went backwards by 10s, and wrote 60, 50, 40, 30 on the tick marks.
- They didn't know that the multiunit of 10 has to be the same length everywhere on the line.

LMR © 7

The inset at the bottom of this page also illustrates the LMR technique of "Pushing Student Thinking" by introducing a hypothetical student who expresses one of the common partial understandings.

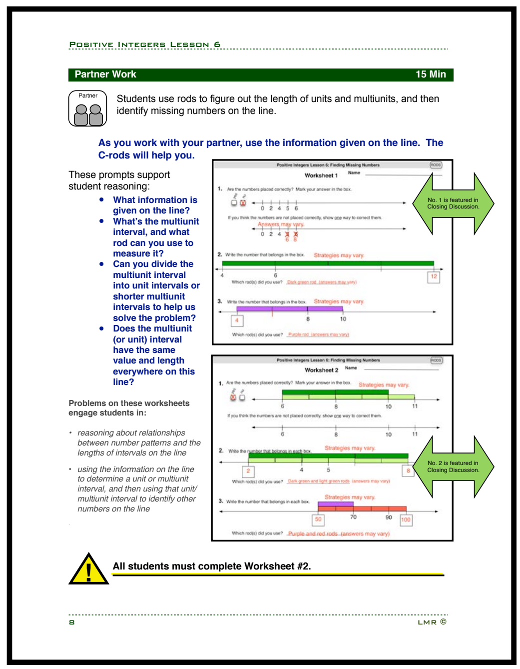


Partner Work: In the most common format for Partner Work, students work in parallel on a sequence of worksheets as partners discuss their interpretations of the tasks and their strategies for solution. The teacher roves to push student thinking and provide support.To ensure that all students have the opportunity to work with the core mathematics of the lesson, all students complete the first one or two worksheets as indicated in the guide. Those students with more advanced understandings can continue with additional worksheets featuring tasks on the same mathematical ideas but with more challenging values.

Alternative Partner Work formats are used in some lessons. Formats include small group work outdoors and a 'human number line' activity indoors.

Lesson Guide features include:

- answer keys (that note that students' correct answers and strategies may vary)
- green arrows to indicate tasks that are the focus of the Closing Discussion
- suggested prompts in blue text
- a yellow caution icon to indicate which worksheets all students must complete prior to the Closing Discussion.



Positive Integers Lesson 6

Partner Work — **15 Min**

Partner

Students use rods to figure out the length of units and multiunits, and then identify missing numbers on the line.

As you work with your partner, use the information given on the line. The C-rods will help you.

These prompts support student reasoning:

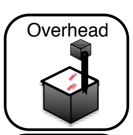
- **What information is given on the line?**
- **What's the multiunit interval, and what rod can you use to measure it?**
- **Can you divide the multiunit interval into unit intervals or shorter multiunit intervals to help us solve the problem?**
- **Does the multiunit (or unit) interval have the same value and length everywhere on this line?**

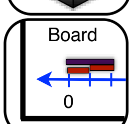
Problems on these worksheets engage students in:

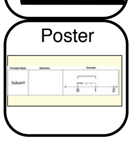
- *reasoning about relationships between number patterns and the lengths of intervals on the line*
- *using the information on the line to determine a unit or multiunit interval, and then using that unit/ multiunit interval to identify other numbers on the line*

All students must complete Worksheet #2.

8 LMR ©

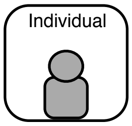




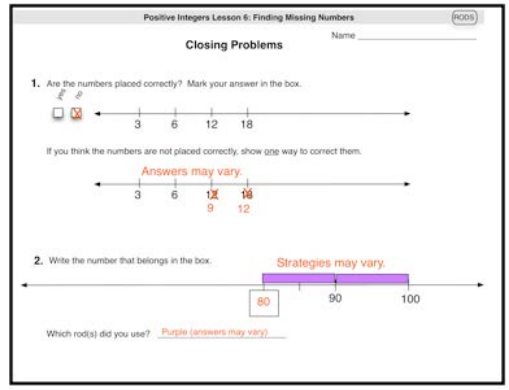


Closing Discussion: The Closing Discussion is designed to help students consolidate their understandings of the lesson. The guide features are the same as the features for the Opening Discussion.

.



Closing Problems: Students conclude most lessons with independent work on problems that mirror the Opening Problems. The guide provides an answer key to facilitate formative assessment. Teachers review the responses to identify students who need additional support.



Homework is an opportunity for students to continue work on the day's problems, and the work can be reviewed the following day. The guide provides an answer key (with the note that correct answers and strategies may vary).

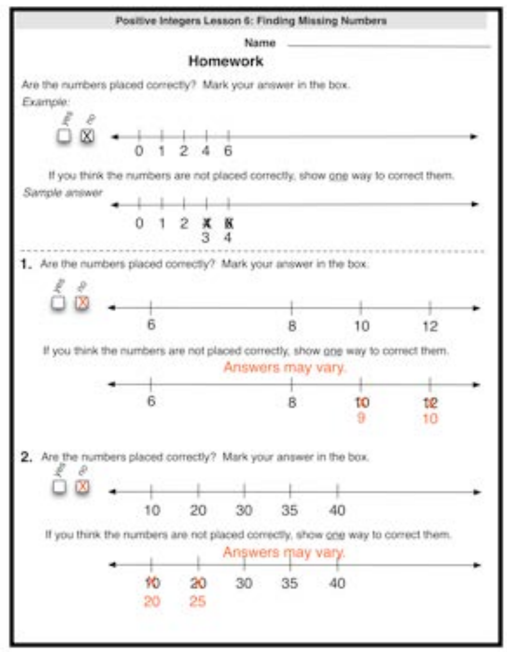


Table 2 on the following page lists each lesson by title, the principles & definitions introduced, and approximate lesson time.

Table 2. Lessons, Principles/Definitions, and Timing

	Lesson	Title	New Principles & Definitions	Time
Positive Integers	Pos Integers 1	Cuisenaire Rod Relationships		35 mins
	Pos Integers 2	Introduction to Number Lines	Order 0 is a number	60 mins
	Pos Integers 3	Unit Intervals	Interval Unit interval	55 mins
	Pos Integers 4	Multiunit Intervals	Multiunit Interval Every number has a place	60 mins
	Pos Integers 5	Coordinating Unit and Multiunit Intervals		50 mins
	Pos Integers 6	Finding Missing Numbers		55 mins
	Pos Integers 7	Using Informal Measurement Tools		60 mins
	Pos Integers 8	Positive Integers Review		35 mins
Negative Integers	Neg Integers 1	Introduction to Negative Numbers	Symmetry	50 mins
	Neg Integers 2	Using Symmetry		45 mins
	Neg Integers 3	Symmetry and Estimation		45 mins
	Neg Integers 4	Ordering and Comparing		50 mins
Fractions	Fractions 1	Subunit Intervals	Subunit	55 mins
	Fractions 2	Defining Denominator, Numerator, and Fraction	Denominator Numerator Fraction	45 mins
	Fractions 3	Labeling Fractions and Understanding Lengths of Subunits	Length of subunit	50 mins
	Fractions 4	Fractions Less than 1 - Measuring Lengths		35 mins
	Fractions 5	Reasoning about Fractions Less than 1		50 mins
	Fractions 6	Measuring Distances Less than 1		40 mins
	Fractions 7	Mixed Numbers	Mixed number	55 mins
	Fractions 8	Fractions Greater than 1	Order Whole numbers as fractions	60 mins
	Fractions 9	Introduction to Equivalent Fractions	Equivalent fractions	55 mins
	Fractions 10	Equivalent Fractions - More Strategies		55 mins
	Fractions 11	Which Fractions are Equivalent?		55 mins
	Fractions 12	Order and Compare with Benchmarks	Benchmarks	55 mins
	Fractions 13	Ordering and Comparing		flexible
	Fractions 14	Fractions Review		flexible